

Roots of polynomials Cheat Sheet

This chapter is concerned with identifying the relationship between the roots of quadratic, cubic and quartic polynomials.

Roots of a quadratic equation

A quadratic equation could have 2 real roots, or 2 complex roots.

- If α and β are roots of the equation $ax^2 + bx + c = 0$, then:

$$\Rightarrow \alpha + \beta = -\frac{b}{a}$$

$$\Rightarrow \alpha\beta = \frac{c}{a}$$

Roots of a cubic equation

A cubic equation could have 3 real roots or 1 real root and 2 complex roots.

- If α, β and γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, then:

$$\Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}$$

$$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\Rightarrow \alpha\beta\gamma = -\frac{d}{a}$$

Roots of a quartic equation

A quartic equation could have 4 real roots, 4 complex roots or 2 real and 2 complex roots.

- If α, β, γ and δ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then:

$$\Rightarrow \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$\Rightarrow \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\Rightarrow \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$$

$$\Rightarrow \alpha\beta\gamma\delta = \frac{e}{a}$$

You can use the following abbreviations to simplify things:

$$\Sigma \alpha = -\frac{b}{a}, \Sigma \alpha\beta = \frac{c}{a}, \Sigma \alpha\beta\gamma = -\frac{d}{a}$$

Expressions relating to the roots of a polynomial

You can use the following rules to quickly find the values of some expressions concerning the roots of a polynomial:

- Reciprocals:

Quadratic: $\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

Cubic: $\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$

Quartic: $\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta}$

You may decide not to remember these results since they are quite easy to prove; in each case you simply need to combine the fractions to achieve the RHS.

- Products of powers:

Quadratic: $\Rightarrow \alpha^n \times \beta^n = (\alpha\beta)^n$

Cubic: $\Rightarrow \alpha^n \times \beta^n \times \gamma^n = (\alpha\beta\gamma)^n$

Quartic: $\Rightarrow \alpha^n \times \beta^n \times \gamma^n \times \delta^n = (\alpha\beta\gamma\delta)^n$

In general, you can remember that $\Sigma \alpha^2 = (\Sigma \alpha)^2 - 2(\Sigma \alpha\beta)$

- Rules for sums of squares:

Quadratic: $\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

Cubic: $\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

Quartic: $\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

- Rules for sums of cubes:

Quadratic: $\Rightarrow \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

Cubic: $\Rightarrow \alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$

You won't need to know the result for a quartic polynomial.

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Example 1: The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = \frac{-1+i}{2}$ and $\beta = \frac{-1-i}{2}$. Find integer values for a, b and c .

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| Using $\alpha + \beta = -\frac{b}{a}$ | $\alpha + \beta = \frac{-1+i}{2} + \frac{-1-i}{2} = -1 = -\frac{b}{a}$ |
| Simplifying: | $\therefore a = b$ |
| Using $\alpha\beta = \frac{c}{a}$ | $\alpha\beta = \left(\frac{-1+i}{2}\right)\left(\frac{-1-i}{2}\right) = \frac{1}{2} = \frac{c}{a}$ |
| Simplifying: | $\therefore \frac{1}{2}a = c$ |
| Rewriting the quadratic with $b = a$ and $c = \frac{1}{2}a$: | $ax^2 + ax + \frac{1}{2}a = 0$ $x^2 + x + \frac{1}{2} = 0$ |
| Dividing through by a : | |
| Multiplying by 2 to ensure that the constant is an integer: | $2x^2 + 2x + 1 = 0$ $a = 2, b = 2, c = 1$ |

Example 2: The equation $mx^2 + 4x + 4m = 0$ has roots of the form k and $2k$. Find the values of m and k .

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| Using $\alpha + \beta = -\frac{b}{a}$: | $\alpha + \beta = k + 2k = 3k = -\frac{4}{m}$ |
| Simplifying: | $\therefore m = -\frac{4}{3k}$ |
| Using $\alpha\beta = \frac{c}{a}$: | $\alpha\beta = (k)(2k) = 2k^2 = \frac{4m}{m} = 4$ |
| Simplifying and solving for k : | $\therefore k^2 = 2$ So $k = \pm\sqrt{2}$ |
| We have two sets of solutions because k has two possible values. Use $m = -\frac{4}{3k}$ to find the corresponding value of m in each case. | If $k = \sqrt{2}, m = -\frac{4}{3\sqrt{2}} = -\frac{2\sqrt{2}}{3}$ If $k = -\sqrt{2}, m = \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}$ |

Example 3: α, β and γ are roots of the cubic equation $7x^3 - 4x^2 - x + 6 = 0$. Find the values of:

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| (a) $\alpha + \beta + \gamma$ | (b) $\alpha\beta\gamma$ | (c) $\alpha^3\beta^3\gamma^3$ | (d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ |
| Using $\alpha + \beta + \gamma = -\frac{b}{a}$: | $\alpha + \beta + \gamma = -\frac{-4}{7} = \frac{4}{7}$ | | |
| Using $\alpha\beta\gamma = -\frac{d}{a}$: | $\alpha\beta\gamma = -\frac{6}{7}$ | | |
| Using $\alpha\beta = \frac{c}{a}$: | $\alpha^3\beta^3\gamma^3 = (\alpha\beta\gamma)^3 = \left(-\frac{6}{7}\right)^3 = -\frac{216}{343}$ | | |
| Using the reciprocal result for cubics: | $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$ | | |
| Using $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$: | $\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{1}{7}$ | | |
| Substituting back into the reciprocal result: | $\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{-\frac{1}{7}}{-\frac{6}{7}} = \frac{1}{6}$ | | |

Example 4: The roots of the equation $x^4 + 2x^2 - x + 3 = 0$ are α, β, γ and δ .

(a) Write down the values of $\Sigma \alpha, \Sigma \alpha\beta, \Sigma \alpha\beta\gamma$ and $\Sigma \alpha\beta\gamma\delta$.

(b) Hence find the values of:

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ (ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ (iii) $(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$

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| Using $\Sigma \alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$: | $\alpha + \beta + \gamma + \delta = 0$ |
| Using $\Sigma \alpha\beta = \frac{c}{a}$ | $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{2}{1} = 2$ |
| Using $\Sigma \alpha\beta\gamma = -\frac{d}{a}$ | $\Sigma \alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{-1}{1} = 1$ |
| Using $\alpha\beta\gamma\delta = \frac{e}{a}$ | $\Sigma \alpha\beta\gamma\delta = \alpha\beta\gamma\delta = \frac{3}{1} = 3$ |
| $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta}$ | $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\Sigma \alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{1}{3}$ |
| Using $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$ | $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (0)^2 - 2(2) = -4$ |

Linear transformations of roots

Given a polynomial (up to the fourth degree), you need to be able to find the equation of a second polynomial whose roots are a linear transformation of the roots of the first.

- If a polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ has roots α, β, γ and δ then the polynomial with roots $g\alpha + h, g\beta + h, g\gamma + h$ and $g\delta + h$, where g and h are real constants, is given by $f\left(\frac{w-h}{g}\right)$.

The same logic follows if the polynomial is cubic or quadratic.

Example 5: The quartic equation $2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$ has roots α, β, γ and δ . Find equations with integer coefficients that have roots:

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| (i) If $f(x) = 2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$ then the new equation will be given by $f\left(\frac{w}{2}\right)$. | $f\left(\frac{w}{2}\right) = 2\left(\frac{w}{2}\right)^4 + 4\left(\frac{w}{2}\right)^3 - 5\left(\frac{w}{2}\right)^2 + 2\left(\frac{w}{2}\right) - 1 = 0$ |
| Simplifying: | $\Rightarrow \frac{1}{8}w^4 + \frac{1}{2}w^3 - \frac{5}{4}w^2 + w - 1 = 0$ |
| Multiplying by 8 to ensure all coefficients are integers: | $\Rightarrow w^4 + 4w^3 - 10w^2 + 8w - 8 = 0$ |
| (ii) If $f(x) = 2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$ then the new equation will be given by $f(w + 1)$. | $f(w + 1) = 2(w + 1)^4 + 4(w + 1)^3 - 5(w + 1)^2 + 2(w + 1) - 1 = 0$ |
| Expanding brackets then simplifying: | $\Rightarrow 2(w^4 + 4w^3 + 6w^2 + 4w + 1) + 4(w^3 + 3w^2 + 3w + 1) - 5(w^2 + 2w + 1) + 2w + 2 - 1 = 0$ $\Rightarrow 2w^4 + 12w^3 + 19w^2 + 12w + 2 = 0$ |